Estimating Flexible Functional Forms Using Macroeconomic Data

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Why is it Important to Estimate *Aggregate Production Functions* or their *Dual Representations*?

- Estimates for dual representations of aggregate technology sets are important for a number of reasons:

- (1) They can generate estimates of ***aggregate technical progress***;
- (2) They can generate estimates of the ***biases of technical change*** and
- (3) They lead to estimates for various ***demand and supply elasticities***.

- Index number methods are available for measuring ***Total Factor Productivity*** and ***technical progress***, as are ***nonparametric methods***, but these methods cannot estimate ***elasticities or biases in technical change***.

Should we Estimate *Gross Output or GDP Functions* or *Joint Cost Functions*?

- If we are using ***national accounts data*** to measure technology sets, then we argue that it is better to estimate ***joint cost functions*** rather than ***gross output*** (or *GDP*) ***functions***. We define these functions as follows.
  - Let ***S*** be the technology set of a production unit. We assume that ***S*** is a nonempty, closed cone that exhibits free disposal of inputs and outputs. ***S*** is the set of nonnegative feasible output vectors $\mathbf{y} \equiv [y_1,\ldots,y_M]^T$ that can be produced by nonnegative input vectors $\mathbf{x} \equiv [x_1,\ldots,x_N]^T$.
    - Notation: y and x are column vectors. The transpose of y and x are y^T and x^T.
  - Suppose the production unit faces the strictly positive output price vector $\mathbf{p} \equiv [p_1,\ldots,p_M]^T$ and the strictly positive input price vector $\mathbf{w} \equiv [w_1,\ldots,w_N]^T$.
  - The ***gross output function***, **G(p,x)** for this production unit is defined as follows:

(1) $G(p,x) \equiv \max_y \{p^T y : (y,x) \in S\}$.

- Alternative names for this function are the ***national product function*** Samuelson (1953; 10), the ***gross profit function*** Gorman (1968), the ***conditional profit function*** McFadden (1966) (1978), the ***variable profit function*** Diewert (1973), the ***GDP function*** Kohli (1978) (1991).
- Thus **G(p,x)** is the maximum revenue the production unit can generate if it faces output prices **p** and uses the input vector **x** to produce the ***revenue maximizing*** output **y** which solves the constrained ***optimization problem***.
- If **intermediate inputs** are included in the vector **y** (indexed by negative signs), then **G(p,x)** is a ***value added function*** or at the national level, it is a **GDP function**. The properties of this function are studied by McFadden (1966) (1978), Diewert (1973) (1974) (2018) and others.
  - Note that we are assuming constant returns to scale in production so that **G(p,x)** is linearly homogenous in p for fixed x and is linearly homogeneous in x for fixed p. If **G(p,x)** is differentiable at a point p, x, **Hotelling's Lemma** (1932; 594) implies that the vector of ***output supply functions*** regarded as functions of **p** and **x**, **y(p,x)**, can be obtained by differentiating **G(p,x)** with respect to the components of **p**:

(2) $y(p,x) = \nabla_p G(p,x)$.

- Samuelson's Lemma (1953; 10) (see also Diewert (1974; 140)) implies that the producer's system of inverse input demand functions regarded as functions of **p** and **x**, **w(p,x)**, can be obtained by differentiating **G(p,x)** with respect to the components of **x**:

  (3) $w(p,x) = \nabla_x G(p,x)$.

- Instead of conditioning on output prices **p** and input quantities **x**, the production unit's ***joint cost function*, C(y,w),** is defined as the ***minimum cost*** of producing a given output vector **y**, and hence conditions on input prices **w** and the output quantities **y**:

  (4) $C(y,w) \equiv \min_x (w^T x : (y,x) \in S\}$.

  – Under our strong regularity conditions on the set S, it can be shown that C(y,w) is linearly homogeneous in the components of y holding w constant and is linearly homogeneous in the components of w holding y constant; see for example, Diewert (2018).

- If **C(y,w)** is differentiable at a point **y**,**w**, then differentiating **C(y,w)** with respect to the components of **y** will generate the ***vector of marginal costs***.
- If the producer takes output prices as being fixed and there are competitive markets, then this ***vector of marginal costs*** will be equal to the ***vector of selling prices* p**.
- Thus the production unit's system of **inverse output supply functions, p(y,w)**, can be obtained by differentiating **C(y,w)** with respect to the components of **y**:

  (5) $p(y,w) = \nabla_y C(y,w)$.

- **Shephard's Lemma** (1953) implies that the production unit's system of ***input demand functions*** regarded as functions of **y** and **w**, **x(y,p)**, can be obtained by differentiating **C(y,w)** with respect to the components of **w**:

  (6) $x(y,w) = \nabla_w C(y.w)$.

- Thus we have two alternative methods for estimating a representation of the technology set ***S***:

- The first representation assumes a functional form for the ***gross output function***, **G(p,x)**, and uses equations (2) and (3) as estimating equations.

- The second representation assumes a functional form for the ***joint cost function***, **C(y,w)**, and uses equations (5) and (6) as estimating equations.

Question:

- **Can we choose which of these two representations is "*best*"?**

Use of the *Joint Cost Function* as the Representation of Technology

- Consider the specialization of the general joint cost function **C(y,w)** defined by (4):

(7) $C(y,w) \equiv \Sigma_{m=1}^{M} c^m(w)y_m$.

- The function $\mathbf{c^m(w)}$ is the unit cost function that is dual to the *single output constant returns to scale production function* for sector **m**, $y_m = f^m(x^m)$ for $m = 1,\ldots M$ where x^m is the vector of inputs used by sector m.
- This is the small open country production framework considered by Samuelson (1953) in his seminal paper.
  - Note that the aggregate input vector x is equal to $\Sigma_{m=1}^{M} x^m$ and the mth output price is equal to $p_m = c^m(w)$ for $m = 1,\ldots,M$.
  - Note also that by applying Shephard's Lemma to each sector, we can deduce that $x^m = \nabla_w c^m(w) y_m$ for $m = 1,\ldots,M$.

- Making use of these equalities, differentiate both sides of (7) with respect to the components of w. We obtain:

(8) $\nabla_w C(y.w) = \Sigma_{m=1}^{M} \nabla_w c^m(w) y_m = \Sigma_{m=1}^{M} x^m \equiv x.$

- **Now further specialize the unit cost functions $c^m(w)$ to be linear functions of w:**

(9) $c^m(w) \equiv \Sigma_{n=1}^{N} w_n d_{nm}$; $m = 1,\ldots,M$

where the d_{nm} are NM constants.

This means that the sectoral production functions are ***Leontief (no substitution) production functions***.

- Define the N by M matrix of the d_{nm} as **$C \equiv [d_{nm}]$.** Substitute definitions (9) into (7) and we obtain the expression for the ***overall joint cost function***:

(10) $C(y,w) = \Sigma_{m=1}^{M} \Sigma_{n=1}^{N} w_n d_{nm} y_m = w^T D y.$

- Thus **the joint cost function for this very special case of *Leontief sectoral production functions*** turns out to be a bilinear form in the vectors of input prices w and of output quantities y.
- Equations (5) and (6) for this special case turns out to be the following estimating equations:

(11) $p^t = D^T w^t$;

(12) $x^t = D y^t$

where p^t, w^t, y^t and x^t are the period t vectors of observed prices and quantities.

- The estimating equations defined by (11) and (12) are linear in the **unknown MN parameters** but of course, there are cross equation equality restrictions.
  - However, this model can readily be estimated using standard nonlinear regression packages like SHAZAM.

Use of the *Gross Output or GDP Function* as the Representation of Technology

- Consider the alternative specializations of the general gross output function **G(p,x)** defined by (1):

  (13) $G(p,x) \equiv \Sigma_{m=1}^{M} g^m(x)p_m$;

  (14) $G(p,x) \equiv \Sigma_{n=1}^{N} h^n(p)x_n$;

  (15) $G(p,x) \equiv \Sigma_{m=1}^{M} \Sigma_{n=1}^{N} x_n e_{nm} p_m = x^T E p$

  where $E \equiv [e_{nm}]$ is an **N by M matrix of constants**.
- The production model defined by (13) implies that output $\mathbf{y_m}$ is equal to the function $\mathbf{g^m(x)}$ of aggregate input **x** for each output m = 1,…,M.

- In the context where outputs are produced by sectoral production functions, ***this model is not very sensible***: the output of sector **m** is produced by the sector **m** vector of inputs, $\mathbf{x^m}$; not by the aggregate input vector **x**.

(14) $G(p,x) \equiv \Sigma_{n=1}^{N} h^n(p)x_n$;

(15) $G(p,x) \equiv \Sigma_{m=1}^{M} \Sigma_{n=1}^{N} x_n e_{nm} p_m = x^T E p$

– The production model defined by (14) implies that each unit of aggregate input **n, x_n**, produces the vector of outputs **$\nabla_p h^n(p)$** independently of all other inputs. This is also ***not a very sensible assumption***.

- Hence the bilinear model defined by (15), which is a special case of the models defined by (13) and (14), is also ***not a sensible model of production*** in the sectoral production function context.

Should we Estimate *Gross Output Functions* or *Joint Cost Functions*?

- When we are estimating ***Gross Output Functions***, it proves to be convenient to start by estimating the bilinear function defined by (15), **$G(p,x) = x^TEp$**.
- When we are estimating ***Joint Cost functions***, it is convenient to start by estimating the bilinear function defined by (10), **$C(y,w) = w^TDy$**.
- If we are estimating ***the technology of a single production unit***, then it may not matter much whether we estimate the technology by estimating a ***gross output function*** or a ***joint cost function***.
- However, the situation is different if we are estimating an aggregate technology that sums over sectoral production possibilities sets and we use the bilinear functions $G(p,x) = x^TEp$ or $C(y,w) = w^TDy$ as representations of the aggregate technology set using only aggregate data.
  - **Estimating the *bilinear gross output function* $G(p,x) = x^TEp$ using aggregate data that has aggregated over *sectors is not going to fit the aggregate data very well* compared to using the *bilinear joint cost function*, $C(y,w) = w^TDy$.**

- If we estimate a national production possibilities set using national accounts data on the **national output aggregates C+G+I+X–M** along with national labour **input L and the various types of capital services, Machinery and Equipment services $K_{M\&E}$, structures K_S and land K_L,** then price and quantity data for these macroeconomic aggregates will have been constructed by ***sector***.
- Thus it will be the case (to some degree of approximation) that the four national outputs, C, G, I and X, will be functions of L^m, $K_{M\&E}{}^m$, $K_S{}^m$, $K_L{}^m$ and imports M^m for m = 1, 2, 3, 4.
- This means that aggregate cost **C(y,w)** will ***decompose into*** the sum of four ***sectoral cost functions***; i.e., equations (7), $C(y,w) \equiv \Sigma_{m=1}^{M} c^m(w)y_m$, will hold (approximately) where M = 4 and w is an input price vector of dimension 5.
- **Thus a starting point for estimating a *flexible functional form* for C(y,w) is to begin with estimating the special case of *sectoral Leontief cost functions* defined by (10)**, which becomes (16) for our particular macroeconomic data set:

(16) $C(y,w) = \Sigma_{m=1}^{4} \Sigma_{n=1}^{5} w_n d_{nm} y_m = w^T Dy.$

- An advantage of (16) as a functional form is that it will represent a **"sensible" technology over the entire data set**. Using (16) as a starting functional form is much preferred to using the ***Gross Output function*** defined by (15) which becomes (17) for our macroeconomic data set:

  (17) $\mathbf{G(p,x)} \equiv \Sigma_{m=1}^{4} \Sigma_{n=1}^{5} x_n e_{nm} p_m = x^T E p$.

- The above algebra explains why we prefer to estimate ***a joint cost function*** **rather than *a gross output function* when working with macroeconomic data**.
- The bilinear starting functional form defined by (16) will provide a much better ***global approximation*** to the ***national technology*** if we are using national macroeconomic data than will be provided by the starting functional form for a gross output function defined by (17).

Flexible Functional Forms for Joint Cost Functions

- A ***flexible functional form*** for a joint cost function **C(y,w)** has *enough free parameters* so that it can provide a second order Taylor series approximation to an arbitrary *twice continuously differentiable joint cost function* **C^*(y,w)** at an arbitrary point (y^*,w^*).
- For our application, we assume **that both C(y,w) and C^*(y,w) are linearly homogeneous in the components of y holding w constant and are linearly homogeneous in the components of w holding y constant.**
- The linear homogeneity assumptions and the assumption of *twice continuous differentiability* imply that **C(y,w)** will be a ***flexible functional form*** if it has enough parameters to satisfy the conditions (18)-(20):

(18) $\nabla_{yw}^2 C(y^*,w^*) = \nabla_{yw}^2 C^*(y^*,w^*)$; MN restrictions;

(19) $\nabla_{yy}^2 C(y^*,w^*) = \nabla_{yy}^2 C^*(y^*,w^*)$; M(M−1)/2 independent restrictions;

(20) $\nabla_{ww}^2 C(y^*,w^*) = \nabla_{ww}^2 C^*(y^*,w^*)$; N(N−1)/2 independent restrictions.

- There are M^2 restrictions in the matrix equation (19) but Young's Theorem in calculus implies that the upper triangle of matrix elements in the matrix of second order partial derivatives of **$C(y^*,w^*)$** is equal to the lower triangle; i.e., $[\nabla_{yy}{}^2\ C(y^*,w^*)]^T = [\nabla_{yy}{}^2\ C(y^*,w^*)]$ and similarly, $[\nabla_{yy}{}^2\ C^*(y^*,w^*)]^T = [\nabla_{yy}{}^2\ C^*(y^*,w^*)]$.
- Thus there are only $M(M+1)/2$ independent restrictions on the second order partial derivatives of $C(y^*,w^*)$ in the matrix equation (19).
- But due to the linear homogeneity of $C(y,w)$ in the components of y, **Euler's Theorem on homogeneous functions** implies the following M restrictions on the **second order partial derivatives of $C(y^*,w^*)$ and $C^*(y^*,w^*)$:**

  (21) $\nabla_{yy}{}^2\ C(y^*,w^*)y^* = 0_M$; $\nabla_{yy}{}^2\ C^*(y^*,w^*)y^* = 0_M$.

- Since the M by M matrices $\nabla_{yy}{}^2\ C(y^*,w^*)$ and $\nabla_{yy}{}^2\ C^*(y^*,w^*)$ are symmetric, equality of the upper diagonal elements in equations (19) plus the 2M equations in (21) will imply equality of all M^2 elements in the matrix equation $\nabla_{yy}{}^2\ C(y^*,w^*) = \nabla_{yy}{}^2\ C^*(y^*,w^*)$.

- Since the M by M matrices ∇_{yy}^{2} $C(y^{*},w^{*})$ and ∇_{yy}^{2} $C^{*}(y^{*},w^{*})$ are symmetric, equality of the upper diagonal elements in equations (19) plus the 2M equations in (21) will imply equality of all M^{2} elements in the matrix equation ∇_{yy}^{2} $C(y^{*},w^{*}) = \nabla_{yy}^{2}$ $C^{*}(y^{*},w^{*})$.

- Similarly, due to the linear homogeneity of C(y,w) in the components of w, Euler's Theorem on homogeneous functions implies the following N restrictions on the second order partial derivatives of **$C(y^{*},w^{*})$** and **$C^{*}(y^{*},w^{*})$:**

(22) ∇_{ww}^{2} $C(y^{*},w^{*})w^{*} = 0_{N}$; ∇_{ww}^{2} $C^{*}(y^{*},w^{*})w^{*} = 0_{N}$.

- Since the N by N matrices ∇_{ww}^{2} $C(y^{*},w^{*})$ and ∇_{ww}^{2} $C^{*}(y^{*},w^{*})$ are symmetric, equality of the upper diagonal elements in equations (20) plus the wN equations in (22) will imply equality of all N^{2} elements in the matrix equation ∇_{ww}^{2} $C(y^{*},w^{*}) = \nabla_{ww}^{2}$ $C^{*}(y^{*},w^{*})$.

The *Normalized Quadratic* Joint Cost Function

- Let $\mathbf{y}^* \equiv [y_1{}^*,\ldots,y_M{}^*]^T >> 0_M$ be a positive reference ***output*** vector and let $\mathbf{w}^* \equiv [w_1{}^*,\ldots,w_N{}^*]^T >> 0_N$ be a positive vector of reference ***input*** prices. Let $\boldsymbol{\alpha} \equiv [\alpha_1,\ldots,\alpha_N]^T >> 0_N$ and $\boldsymbol{\beta} \equiv [\beta_1,\ldots, \beta_M]^T >> 0_M$ be positive vector of **predetermined constants** and that satisfy the linear restrictions (23):

  (23) $\alpha^T w^* = 1$; $\beta^T y^* = 1$.

- The basic ***Normalized Quadratic Joint Cost Function*, C(y,w)**, is defined as follows:

  (24) $\mathbf{C(y,w) \equiv (½)(w^TAw)(\alpha^Tw)^{-1}(\beta^Ty) + (½)(y^TBy)(\alpha^Tw)(\beta^Ty)^{-1} + w^TDy.}$

- This functional form is basically the same as the ***Normalized Quadratic Value Added Function*** **Π(p,x)** that was defined in Diewert and Fox (2021).
- The M by N matrix D is unrestricted but there are some restrictions on the A and B matrices that need to be imposed in order for the estimated joint cost function to satisfy curvature conditions and to be a ***parsimonious flexible functional form*** at the point $\mathbf{y}^*, \mathbf{w}^*$.

- Assume that the matrix A has the following properties:

(25) A is a negative semidefinite N by N matrix;
(26) A is symmetric so that $A = A^T$;
(27) $Aw^* = 0_N$.

- Assume that the matrix B has the following properties:

(28) B is a positive semidefinite M by M matrix;
(29) B is symmetric so that $B = B^T$;
(30) $By^* = 0_M$.

- Now compute the first and second order partial derivatives of **C(y,w)** and evaluate them at the point **(y^*,w^*)**. Using the restrictions (23, (27) and (30), we obtain the following first and second order partial derivatives:

  (31) $\nabla_y C(y^*,w^*) = D^T w^*$;

  (32) $\nabla_w C(y^*,w^*) = Dy^*$;

  (33) $\nabla_{yy}{}^2 C(y^*,w^*) = B$;

  (34) $\nabla_{ww}{}^2 C(y^*,w^*) = A$;

  (35) $\nabla_{yw}{}^2 C^*(y^*,w^*) = D^T$.

- To prove the flexibility of the ***Normalized Quadratic Joint Cost Function*** defined by (24) with the restrictions (25)-(30), we need to find matrices A, B and D that lead to the satisfaction of equations (18)-(20). Using equations (33)-(35), this is very simple: define A, B and D as follows:

  (36) $A \equiv \nabla_{ww}{}^2 C^*(y^*,w^*)$;

  (37) $B \equiv \nabla_{ww}{}^2 C^*(y^*,w^*)$;

  (38) $D \equiv [\nabla_{yw}{}^2 C^*(y^*,w^*)]^T$.

- Under our regularity conditions on the production possibilities set **S**, it can be shown **that $\nabla_{ww}^2 C^*(y^*,w^*)$** is a symmetric negative semidefinite matrix which satisfies $\nabla_{ww}^2 C^*(y^*,w^*)w^* = 0_N$ and hence, the matrix A will satisfy the restrictions (25)-(27).

- It also can be shown that $\nabla_{yy}^2 C^*(y^*,w^*)$ is a symmetric positive semidefinite matrix which satisfies $\nabla_{yy}^2 C^*(y^*,w^*)y^* = 0_N$ and hence, B will satisfy the restrictions (28)-(30).

- This establishes the ***flexibility of the basic Normalized Quadratic Joint Cost function*.**

Estimating Equations

- Our data are based on ***Chinese National Accounts data*** for the years 1970-2020 and are described in Diewert, Nomura and Shimizu (2023).
- The data were constructed by **Koji Nomura** and other researchers associated with the ***Asian Productivity Organization***.

- **The four outputs are C, G, I and X and the 5 inputs are aggregate of Labour, Machinery and Equipment and other capital inputs, Structures, Land and Imports, L, $K_{M\&E}$, K_S, K_L and M.**

- Denote the 4 dimensional output price and quantity vectors for year t by p^t and y^t and the 5 dimensional input price and quantity vectors for year t by w^t and x^t for $t = 1970,\ldots,2021$.

- The **estimating equations** using the **basic Normalized Quadratic Joint Cost function defined** by (24) are the equations for t = 1970,…,2020:

(39) $\mathbf{p^t} = D^Tw^t + (\alpha^Tw^t)(\beta^Ty^t)^{-1}By^t + (½)(w^{tT}Aw^t)(\alpha^Tw^t)^{-1}\beta - (½)(y^{tT}By^t)(\alpha^Tw^t)(\beta^Ty^t)^{-2}\beta$;

(40) $\mathbf{x^t} = Dy^t + (\beta^Ty^t)(\alpha^Tw^t)^{-1}Aw^t + (½)(y^{tT}By^t)(\beta^Ty^t)^{-1}\alpha - (½)(w^{tT}Aw^t)(\beta^Ty^t)(\alpha^Tw^t)^{-2}\alpha$.

- It turns out that Chinese *output prices* $\mathbf{p^t}$ and *input quantities* $\mathbf{x^t}$ have grown enormously over our sample period so the dependent variables in equations (39) and (40) (conditioned on the exogenous variables) ***generate heteroskedastic error terms***.
- This ***heteroskedasticity problem*** can be mitigated if we divide both sides of equations (39) for year t by the *year t input price index* $\boldsymbol{\alpha^Tw^t}$ and divide both sides of equations (40) for year t by the *year t input quantity index* $\boldsymbol{\beta^Ty^t}$.
- Define the ***normalized variables*** for t = 1970,…,2020:

(41) $p^{t*} \equiv p^t/\alpha^Tw^t$; $x^t \equiv x^t/\beta^Ty^t$; $w^{t*} \equiv w^t/\alpha^Tw^t$; $y^{t*} \equiv x^t/\beta^Ty^t$.

- Substitute definitions (41) into (39) and (40) and we obtain the **estimating equations**:

$$(42)\ p^{t*} = D^T w^{t*} + B y^{t*} + (½)(w^{t*T} A w^{t*})\beta - (½)(y^{t*T} B y^{t*})\beta;$$

$$(43)\ x^{t*} = D y^{t*} + A w^{t*} + (½)(y^{t*T} B y^{t*})\alpha - (½)(w^{t*T} A w^{t*})\alpha.$$

  Note that the right hand sides of equations (42) and (43) are linear in the unknown parameters which appear in the matrices A, B and D.

- Of course, there are cross equation equality restrictions that prevent us from simply using equation by equation ordinary least squares to estimate these parameters.
- We used the nonlinear regression option in Shazam (see White (2004)) to estimate the unknown parameters in the above estimating equations.
- Equations (42) are the 4 ***output price equations*** that use the **(normalized) prices of C, G, I and X** as dependent variables.
- Equations (43) are the 5 ***input quantity equations*** that use the (normalized) aggregate input demands for **labour, M&E, Structures, Land and Imports** as dependent variables.

Defining the Exogenous Vectors α and β

- The year t vectors of exogenous variables in the estimating equations (42) and (43) are $\mathbf{y^t}$ and $\mathbf{w^t}$, the year **t** output quantity and input price vectors.
- We choose **units of measurement for the outputs and inputs so that all output quantities and input prices** for 1970 equal one; thus $y^{1970} = 1_4$ and $w^{1970} = 1_5$.
- After this change in the units of measurement, define the ***sample wide average output price vector*** as $\mathbf{p}^* \equiv (1/51)\, \Sigma_{t=1970}^{2020}\, \mathbf{p}^t$ and the ***sample wide average input quantity vector*** as $\mathbf{x}^* \equiv (1/51)\, \Sigma_{t=1970}^{2020}\, \mathbf{x}^t$.
- Define the vectors β and α as the normalizations of $\mathbf{p}^*$ and $\mathbf{x}^*$:

  (44) $\beta \equiv p^*/p^{*T}1_4$; $\alpha \equiv x^*/x^{*T}1_5$.

  – Since $y^{1970} = 1_4$, we see that $\beta^T y^{1970} = 1$ and since $w^{1970} = 1_5$, it follows that $\alpha^T w^{1070} = 1$ as well.

Imposing Curvature Conditions on the Matrices A and B

- We need to ensure that our estimated A matrix is a negative semidefinite symmetric matrix that satisfies $Aw^* = 0_5$ (see (25)-(27) above). We choose our w^* to be w^{1970} which is the vector of ones, 1_5, for our empirical application to China.
- The imposition of symmetry and negative semidefiniteness on A can be done using a technique due to Wiley, Schmidt and Bramble (1973): simply replace the matrix A by:

  (45) $A \equiv - UU^T$

  - where U is a 5 by 5 ***lower triangular matrix***; i.e., $u_{ij} = 0$ if $i < j$.
- The restrictions $Aw^* = A1_5 = 0_5$ on A can be imposed if we impose the following restrictions on U:

  (46) $U^T 1_5 = 0_5$.

- The imposition of symmetry and positive semidefiniteness on B can be accomplished in a similar fashion: set B equal to:

  (47) $B \equiv VV^T$

- where V is a 4 by 4 *lower triangular matrix*; i.e., $v_{ij} = 0$ if $i < j$.
- The restrictions $By^* = B1_4 = 0_4$ on A can be imposed if we impose the following restrictions on V:

  (48) $V^T 1_4 = 0_4$.

- The restrictions (46) and (48) imply that the maximum rank for the A and B matrices is 4 and 3 respectively.
- **Once the matrices A and B in the estimating equations (42) and (43) are replaced by**

  – UU^T and VV^T respectively, the resulting estimating equations are no longer linear in the unknown parameters and a nonlinear regression package must be used.

- For more details on how this nonlinear estimation works, see Diewert and Wales (1987) (1988).

The Problem of *Trending Elasticities*

- Diewert and Lawrence (2002) noted a problem that arises when a normalized quadratic functional form is estimated: the resulting **elasticities of input demand and inverse elasticities of output supply will tend to have strong trends if prices and quantities in the data have strongly divergent trends over the sample period**.
  - Diewert and Lawrence (2002) suggested a method for dealing with this problem: let the components of the A and B (or U and V) matrices have linear time trends over the sample period. We implemented their method for our Chinese data.

- Thus the matrices A and B in the estimating equations (42) and (43) become linear functions of time t, A(t) and B(t).

Adding *Linear Trends* for Cost Saving Technical Progress

- In order to allow for ***cost saving technical progress***, we add the terms **$(a\cdot p)(\beta\cdot x)t$ and $(\alpha\cdot p)(b\cdot x)t$** involving time t to the ***Normalized Joint cost function*** defined by (24).
- The resulting cost function is the **one** where the matrices A and B are **functions of time and adjusted to satisfy curvature conditions** as indicated above:

$$(49)\ C(y,w,t) \equiv (\tfrac{1}{2})(w^TA(t)w)(\alpha^Tw)^{-1}(\beta^Ty) + (\tfrac{1}{2})(y^TB(t)y)(\alpha^Tw)(\beta^Ty)^{-1} + w^TDy + (a\cdot w)(\beta\cdot y)t + (b\cdot y)(\alpha\cdot w)t$$

where $a \equiv [a_1,\ldots,a_5]^T$ and $b \equiv [b_1,\ldots,b_4]^T$ are new parameters which allow for biased technical change. The estimating equations (42) and (43) become:

$$(50)\ p^{t*} = D^Tw^{t*} + B(t)y^{t*} + (\tfrac{1}{2})(w^{t*T}A(t)w^{t*})\beta - (\tfrac{1}{2})(y^{t*T}B(t)y^{t*})\beta + bt + (a^Tw^{t*})t\beta\ ;$$

$$(51)\ x^{t*} = Dy^{t*} + A(t)w^{t*} + (\tfrac{1}{2})(y^{t*T}B(t)y^{t*})\alpha - (\tfrac{1}{2})(w^{t*T}A(t)w^{t*})\alpha + at + (b^Ty^{t*})t\alpha\cdot$$

- In our empirical implementation of the above model, we let time t = 0,1,2,…,51 instead of t = 1970, 1971, …, 2020.
- Thus when t = 0, the new terms at the ends of (50) and (51) vanish.
- Here are the **R^2** for the 9 equations in the model defined by (50) and (51):

0.9875 0.7277 0.6181 0.9265 0.9682 0.1871 0.9929 0.9266 0.4942.

- The above regression fits are not very satisfactory!
- Moreover, if we define year ***t cost saving technical progress*** as $-[\partial C(y^t,w^t,t)/\partial t]/C(y^t,w^t,t)$, we found that the average of these estimates was **2.45%** per year, which is far above the average index number estimate of ***Total Factor Productivity Growth*** that we found in our companion paper, which was **1.20%** per year over the years 1971-2020; see Diewert, Nomura and Shimizu (2023).

The Use of Linear Splines to Model Cost Saving Technical Progress

- Note that the linear terms in time bt appear on the RHS of (50) and the linear terms in time at appear on the RHS of (51) where a and b are technical change parameter vectors.
- **We replaced these linear terms with piece-wise linear functions of time (linear spline functions).** We used the residuals for the model defined by (50) and (51) to determine the break points for the spline functions.
- The number of break points in each equation varied from 5 to 7.
- The final model had **100 parameters** with 51x9 = 459 degrees of freedom.
- Here are the R^2 for the 9 equations:

  0.9983 0.9944 0.9718 0.9958 0.9887 0.9440 0.9980 0.9824 0.9282
- These fits for the 9 equations are satisfactory.

Elasticities of Input Demand and Output Supply

- The ***average rate of cost savings*** due to ***technical progress*** (expressed as a percentage of cost for each year) was **1.34%** per year.
- This is fairly close to our index number estimate of **TFP** for the same data which was **1.20%** per year.

- The average ***own price elasticities of input demand*** were as follows:

Labour: –0.068; M&E: –0.068; Structures: –0.426; Land: –0.175; Imports: –0.222.

- The average ***own inverse elasticities of output supply*** were as follows:

Consumption: 0.1006; Government: 0.0497; Investment: 0.1395; Exports: 0.080.

- However, we used our estimated second order derivatives of the joint cost function to convert the inverse supply elasticities into regular output supply derivatives and then we converted these derivatives into output supply elasticities.

- The ***average own elasticities of output supply*** were as follows for our Chinese data:

  Consumption: 0.7052; Government: 1.6427; Investment: 2.6542; Exports: 1.0390.

Conclusion.

- **It is better to estimate *joint cost functions*** rather than ***gross output functions*** when working with macroeconomic data that aggregates over sectors.
- The ***Normalized Quadratic Joint Cost function*** can be used to model aggregate production possibilities set but the estimation is quite complex.
- A next step in using the methodology outlined in this paper is to generalize it to give us estimates of **markups**.
- Instead of setting **output price equal to marginal cost**, set **output price equal to a markup plus marginal cost**. For an application of this methodology which used a ***translog joint cost function***, see Diewert and Fox (2008).

- **The advantage of the present methodology is that it can better approximate the fact that aggregate data are obtained by aggregating over *sectors* and it is possible to impose curvature conditions globally on the estimated joint cost function**.

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